On a fractional distributed-order oscillator

2005 J. Phys. A: Math. Gen. 38 6703

(http://iopscience.iop.org/0305-4470/38/30/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6703–6713 doi:10.1088/0305-4470/38/30/006

On a fractional distributed-order oscillator

T M Atanackovic[1], **M Budincevic**[2] **and S Pilipovic**[2]

[1] Department of Mechanics, Faculty of Technical Sciences, University of Novi Sad, POB 55, 21121 Novi Sad, Serbia
[2] Department of Mathematics, University of Novi Sad, 21000 Novi Sad, Serbia

Received 23 March 2005, in final form 9 June 2005
Published 13 July 2005
Online at stacks.iop.org/JPhysA/38/6703

Abstract
We consider a viscoelastic rod with a concentrated mass at its end. The mass is moving along the straight line that coincides with the rod axis. The mass is connected by a linear spring and a known active force is acting on it. We assume that the rod is light and described by fractional dissipation. The dynamics of such a system constitutes a problem of a fractional oscillator. In this paper, we shall study some properties of the solutions for the distributed-order fractional derivative viscoelastic rod.

PACS numbers: 46.70.De, 83.60.Bc

1. A viscoelastic rod with concentrated mass at the end

Let $\sigma(x,t)$ and $\varepsilon(x,t)$ be a stress and strain, respectively, in the linear stress state at the point x and at the time instant t. We assume that $x \in [0, l]$ where l is the length of the body. In [3], a constitutive equation was proposed for a linear viscoelastic body in a distributed order form as[3]

$$\int_0^1 \phi_\sigma(\gamma)\sigma^{(\gamma)}(x,t)\,\mathrm{d}\gamma = \int_0^1 \phi_\varepsilon(\gamma)\varepsilon^{(\gamma)}(x,t)\,\mathrm{d}\gamma, \tag{1}$$

where

$$D_t^{(\alpha)}\sigma(x,t) = \sigma^{(\alpha)}(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\sigma(x,t-\xi)\,\mathrm{d}\xi}{\xi^\alpha}, \tag{2}$$

is used to denote α—the derivative of $\sigma(t)$ with respect to time, in the Riemann–Liouville form (see [22]). A special case of (1) is the so-called three-parameter model treated in [10],

[3] In the case when ϕ_σ and (or) ϕ_ε have terms containing the Dirac distribution the integrals in (1) are not Riemann but Stieltjes integrals. Thus, for example $\int_0^1 \sigma_\sigma\delta(\gamma-\alpha)\,\mathrm{d}\gamma$ in (1) should be interpreted as $\int_0^1 \sigma_\sigma\delta(\gamma-\alpha)\,\mathrm{d}\gamma \equiv \int_0^1 \sigma_\sigma\,\mathrm{d}\psi_\sigma(\gamma)$ where $\psi_\sigma(\gamma) = \begin{cases} 0, & 0<\gamma<\alpha \\ 1, & \gamma \geqslant \alpha. \end{cases}$ In general, it is assumed that ψ is a Borel-measurable function of normalized bounded variation.

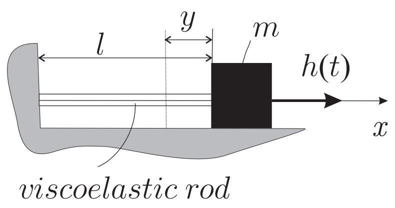

Figure 1. Coordinate system and force configuration.

for example. Many other models of viscoelastic bodies, used earlier, may be considered as a special case of (1). For example, if we take

$$\phi_\sigma = [(\check{\tau}_\sigma)^\gamma \delta(\gamma - \alpha) + \delta(\gamma)], \qquad \phi_\varepsilon = E[(\check{\tau}_\varepsilon)^\gamma \delta(\gamma - \alpha) + \delta(\gamma)], \tag{3}$$

where $\check{\tau}_\sigma$, $\check{\tau}_\varepsilon$ and E are constants and $\delta(\gamma)$ is the Dirac distribution, we obtain

$$\sigma + (\check{\tau}_\sigma)^\alpha \sigma^{(\alpha)} = E[(\check{\tau}_\varepsilon)^\alpha \varepsilon^{(\alpha)} + \varepsilon], \tag{4}$$

i.e., the generalized Zener model of a viscoelastic body (see [2]). Note also that a special case of (1) with

$$\phi_\sigma = \delta(\gamma) + \tau_\sigma \delta(\gamma - \alpha), \qquad \phi_\varepsilon(\gamma) = E[\delta(\gamma) + \tau_\varepsilon \delta(\gamma - \alpha) + \tau_\beta \delta(\gamma - \beta)], \tag{5}$$

leads to the generalized Kelvin–Voigt model that was recently analysed in [21]. For other applications of the distributed-order equations, see [7, 8, 11].

The important problem of the theory of constitutive equations is to formulate the restrictions on the coefficients, or functions, in the constitutive equation that follows from the second law of thermodynamics. Either the internal variable method (see [2] and references given there) or the method based on Fourier transforms (see [6] or [21]) is used. We shall not address this question here (see [4] where the problem was treated).

We assume that $\phi_\sigma(\gamma) = a^\gamma$, $\phi_\varepsilon(\gamma) = \Lambda b^\gamma$. The second law of thermodynamics requires (see [4]) that $\Lambda > 0$, $0 < a < b$. With these functions, (1) becomes

$$\int_0^1 a^\gamma \sigma^{(\gamma)}(x, t)\, \mathrm{d}\gamma = \Lambda \int_0^1 b^\gamma \varepsilon^{(\gamma)}(x, t)\, \mathrm{d}\gamma. \tag{6}$$

The constants a and b have dimensions of time and have the meaning of relaxation times for stress and strain, respectively.

For the case of a linear state of strain, we have $\varepsilon = [\partial u(x, t)/\partial x]$, where $u(x, t)$ is the displacement of the point of the rod at the position x at time instant t. Thus, (6) becomes

$$\int_0^1 a^\gamma \sigma^{(\gamma)}(x, t)\, \mathrm{d}\gamma = \Lambda \int_0^1 b^\gamma \left[\frac{\partial u(x, t)}{\partial x}\right]^{(\gamma)} \mathrm{d}\gamma. \tag{7}$$

A model example of a mechanical system that leads to the equations that we shall study is a light viscoelastic rod of undeformed length l made of material described by (7) and a linear spring with spring constant c. The system is shown in figure 1.

Suppose that the rod is fixed at one end and that at the other end a body B of mass m is fixed. Also, we assume that B moves along the straight line coinciding with the rod axis, and that a prescribed force $h(t)$ is acting on it. The equation of the motion for the rod reads

$$\frac{\partial \sigma(x, t)}{\partial x} = \rho \frac{\partial^2 u(x, t)}{\partial t^2}, \tag{8}$$

subject to

$$u(0,t)=0, \qquad A\sigma(l,t)+m\frac{\partial^2 u(l,t)}{\partial t^2}=h(t). \tag{9}$$

where A is the cross-sectional area of the rod and ρ is the mass density of the rod. Since the rod is light, i.e., the density ρ is zero, we conclude from (8) that $\frac{\partial\sigma(x,t)}{\partial x}=0$, i.e., σ is independent of x.

Let $y(t)=u(l,t)$ be the displacement of the body B from its initial position. Then we assume that the position of the material point that was at x in the undeformed state, in the deformed state is at $x+x\frac{y(t)}{l}$. With this assumption the displacement vector becomes $u=x\frac{y(t)}{l}$, so that the strain $\varepsilon(x,t)=\partial u(x,t)/\partial x=\frac{y(t)}{l}$ is independent of x. Therefore, (7) becomes

$$\int_0^1 a^\gamma \sigma^{(\gamma)}(t)\,\mathrm{d}\gamma=\frac{\Lambda}{l}\int_0^1 b^\gamma y^{(\gamma)}(t)\,\mathrm{d}\gamma. \tag{10}$$

The equation of motion for the body B reads

$$my^{(2)}(t)+\frac{c}{l}y(t)+A\sigma(t)=h(t), \tag{11}$$

Equation (11) is of the type

$$y^{(2)}(t)+\omega^2 y+\beta\sigma(t)=h(t), \tag{12}$$

where $\omega^2=\frac{c}{ml}$ and (see 10)

$$\int_0^1 a^\gamma \sigma^{(\gamma)}(t)\,\mathrm{d}\gamma=\lambda\int_0^1 b^\gamma y^{(\gamma)}(t)\,\mathrm{d}\gamma, \tag{13}$$

and $\beta>0, \lambda>0$. The forcing function $h(t)$ is assumed to be known. To (12), we adjoin the following initial conditions:

$$y(0)=y_0, \qquad y^{(1)}(0)=v_0. \tag{14}$$

We shall treat the fractional oscillator described by (12)–(14). For other treatments of fractional oscillators see, for example, [5, 9, 14, 17–19].

2. Some properties of the solution to (12)–(14)

The formal use of the Laplace transform to system (12)–(14) shows that the framework for solving this system is in fact the space $\mathcal{K}'_+$. It will be shown that $\sigma\in\mathcal{K}'_+$ and $y\in\mathcal{S}'_+$. Recall, $\mathcal{K}'_+$ is the space of exponentially bounded distributions (cf [1, 16]) supported by $[0,\infty)$. This space is related to the well-known space $\mathcal{S}'_+$ [24] of tempered distributions supported by $[0,\infty)$ via: $f\in\mathcal{K}'_+$ if and only if $f=\mathrm{e}^{kx}F$, for some $k\geqslant 0$ and some $F\in\mathcal{S}'_+$.[4] The framework of $\mathcal{K}'_+$ enables the use of the Laplace transform. We give brief explanations concerning these spaces in the appendix.

Formally, applying the Laplace transform to (13) we obtain

$$\frac{as-1}{\ln(as)}\widehat{\sigma}(s)=\lambda\frac{bs-1}{\ln(bs)}\widehat{y}(s), \qquad \mathrm{Re}\, s>0, \tag{15}$$

where $\widehat{y}(s)=\mathcal{L}(y)(s)=\int_0^\infty \mathrm{e}^{-st}y(t)\,\mathrm{d}t$, $\mathrm{Re}\,s>0$, is the Laplace transform of y. Actually, we have to prove that y and σ are elements of $\mathcal{K}'_+$ and to interpret the above integral form of

[4] Note that $\mathcal{S}'_+\subset\mathcal{K}'_+$.

the Laplace transform in the sense of exponential distributions. We will do this later and this implies that the procedure which is to follow is legitimate. From (15), we obtain ($\mathrm{Re}\, s > 0$)

$$\widehat{\sigma}(s) = \lambda \frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1} \widehat{y}(s). \tag{16}$$

Applying now the Laplace transform to (12) and using (16), we get

$$\widehat{y}(s) = \frac{y_0 s + v_0 + \widehat{h}(s)}{F(s)}, \tag{17}$$

where

$$F(s) = s^2 + \omega^2 + \lambda\beta \frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1}. \tag{18}$$

Proposition 1. *For the system (12), (13) we have $y \in \mathcal{S}'_+$ and $\sigma \in \mathcal{K}'_+$.*

Proof. In order to prove that $y \in \mathcal{S}'_+$, we need an estimate of $J(s)$ where

$$J(s) = (s^2 + \omega^2)(\ln bs)(as-1) + \lambda\beta(\ln as)(bs-1), \qquad \mathrm{Re}\, s > 0. \tag{19}$$

There exists $C > 0$ such that

$$|J(s)| > C, \qquad \mathrm{Re}\, s > 0. \tag{20}$$

So let us prove (20). Note that if $\varepsilon > 0$ is small enough, then there exists $d > 0$ such that

$$(|s| \leqslant \varepsilon)(\mathrm{Re}\, s > 0) \Rightarrow (|J(s)| > d). \tag{21}$$

This is a consequence of the fact that $\lim_{|s|\to 0} = |\omega^2 + \lambda\beta|$.

Further, for any $R > 0$ there exists $d_1 > 0$ such that

$$(|s| \in [\varepsilon, R])(\mathrm{Re}\, s > 0) \Rightarrow (|J(s)| > d_1). \tag{22}$$

We can enlarge R so that

$$|s^2 - \omega^2| \geqslant R^2 - |\omega^2|, \qquad |as-1||\ln bs| \geqslant \tfrac{1}{2}|as||\ln bs|.$$

With this and $|\ln bs| = |\ln R + \ln b + \mathrm{i} \arg s| \geqslant (\ln R + \ln b - \pi/2)$, we have

$$|J(s)| \geqslant \frac{aR}{2}(R^2 - |\omega^2|)(\ln b + \ln R - \pi/2) - |\lambda\beta|(bR(\ln a + \ln R + \pi/2) + 1) \geqslant R. \tag{23}$$

Now by (21)–(23), the proof of (20) follows.

For the next estimate, we use the elementary inequality for a polynomial of order 2 and $\ln|s| \leqslant |s| + 1/|s|$, $s \in \mathbb{C}$. By (20), it follows that there exists (another) $C > 0$ such that

$$\begin{aligned} |\widehat{y}(s)| &= \left| \frac{(y_0 s + v_0 + \widehat{h}(s))(\ln bs)(as-1)}{J(s)} \right| \\ &\leqslant C(1+|s|)^2(|\widehat{h}(s)| + 1)\left(\frac{1}{|s|} + |s|\right). \end{aligned} \tag{24}$$

(Note, $h \in \mathcal{S}'_+$.) Recall the characterization of $\mathcal{S}'_+$ (see [24]):

$$g \in \mathcal{S}'_+ \Leftrightarrow |\widehat{g}(s)| \leqslant C \frac{|\mathrm{Re}\, s|^p}{(1 + |\mathrm{Im}\, s|^q)} \qquad \text{for some } C > 0 \quad \text{and} \quad p, q \in \mathbb{R}, \tag{25}$$

for h, (23) and (24), it follows that $\widehat{y}$ satisfies (25) with another constant and thus, $y \in \mathcal{S}'_+$.

We will show that $\sigma \in \mathcal{K}'_+$ by explicit calculation.

Denote

$$\phi_1 = \mathcal{L}^{-1}\left(\frac{\ln as}{\ln bs}\right), \qquad \phi_2 = \mathcal{L}^{-1}\left(\frac{bs-1}{as-1}\right),$$

where the Laplace transformation is taken in the sense of tempered distributions. We have

$$\phi_1 = \mathcal{L}^{-1}\left(1 + \frac{\ln a - \ln b}{\ln b + \ln s}\right) = \delta + \ln\frac{a}{b}\mathcal{L}^{-1}\left(\frac{1}{\ln b + \ln s}\right),$$

where δ is the delta distribution. Note that $\mathcal{L}(b^{-t})(s) = (s + \ln b)^{-1}$, $\operatorname{Re} s > 0$. This and [13], (29), p 132, imply

$$\mathcal{L}^{-1}\left(\frac{1}{\ln b + \ln s}\right)(t) = \int_0^\infty \frac{t^{u-1}\,\mathrm{e}^{-u\ln b}}{\Gamma(u)}\,\mathrm{d}u, \qquad t > 0.$$

Here, we note that for some $k > 0$ and $C > 0$, which depend on b,

$$\left|\int_0^\infty \frac{t^{u-1}\,\mathrm{e}^{-u\ln b}}{\Gamma(u)}\,\mathrm{d}u\right| \leqslant C\,\mathrm{e}^{kt}, \qquad t > 0.$$

and that this integral is not polynomially bounded. Thus, this integral defines an element of $\mathcal{K}'_+\backslash\mathcal{S}'_+$.

We have

$$\phi_1(t) = \delta(t) + \ln\frac{a}{b}\int_0^\infty \frac{t^{u-1}b^{-u}}{\Gamma(u)}\,\mathrm{d}u, \qquad \phi_2(t) = \frac{b}{a}\delta(t) + \frac{b-a}{a^2}\,\mathrm{e}^{t/a}, \qquad t > 0, \tag{26}$$

and both distributions are equal to 0 on $(-\infty, 0)$. Clearly, both distributions are elements of $\mathcal{K}'_+$. The same holds for

$$\sigma(t) = \lambda\phi_1 * \phi_2 * y(t) \tag{27}$$

since $\mathcal{K}'_+$ is a commutative and associative algebra under convolution. □

For the function $F(s)$ we have the following:

Proposition 2. *Let F_0 be the principal branch of F, i.e. the branch for which $\ln z = \ln|z| + \mathrm{i}\arg z$, where $|\arg z| < \pi$. If $\lambda\beta > 0$ and $b > a > 0$, then F_0 has exactly two zeros which are simple, conjugate and placed in the open left half-plane.*

Proof. Clearly,

$$F_0(\bar{s}) = \overline{F_0(s)}, \qquad s \in \mathbb{C} \tag{28}$$

and this implies that $F_0(\overline{s_0}) = 0$ if $F_0(s_0) = 0$. Let us prove that F_0 has exactly two zeros. We will prove this with the argument principle. Consider the domain

$$\Omega = \{s \in \mathbb{C} : 0 < r < |s| < R, |\arg s| < \pi\}$$

and let $C_{r,R}$ be its boundary.

Put $s = R\,\mathrm{e}^{\mathrm{i}t}$, $0 \leqslant t \leqslant \pi$. Then $F_0(R) > 0$ and

$$F_0(R\,\mathrm{e}^{\mathrm{i}t}) = R^2\,\mathrm{e}^{2\mathrm{i}t}\left(1 + \frac{\omega^2}{R^2\,\mathrm{e}^{2\mathrm{i}t}} + \frac{\beta\lambda}{R^2\,\mathrm{e}^{2\mathrm{i}t}}\,\frac{\ln aR + \mathrm{i}t}{\ln bR + \mathrm{i}t}\,\frac{bR\,\mathrm{e}^{\mathrm{i}t} - 1}{aR\,\mathrm{e}^{\mathrm{i}t} - 1}\right).$$

As the expression in the parentheses tends to 1 as $R \to \infty$, we conclude that on the semicircle $s = R\,\mathrm{e}^{\mathrm{i}t}$; $0 \leqslant t \leqslant \pi$.

$$\lim_{R\to\infty} \Delta \operatorname{Arg} F_0(s) = 2\pi, \tag{29}$$

where Δ denotes the variation.

Put $s = t\,\mathrm{e}^{\mathrm{i}\pi}$, $r \leqslant t \leqslant R$. The imaginary part of $F_0(s)$ satisfies

$$\operatorname{Im} F_0(t\,\mathrm{e}^{\mathrm{i}\pi}) = \lambda\beta\frac{bt+1}{at+1}\,\frac{\pi(\ln b - \ln a)}{\pi^2 + \ln^2 bt} > 0$$

and tends to 0 as $t \to 0$ or $t \to \infty$. Thus, we conclude that on this part of the ray (if Δ denotes a variation)

$$\lim_{r\to 0, R\to\infty} \Delta \mathrm{Arg}\ F_0(s) = 0. \tag{30}$$

Put $s = r\,\mathrm{e}^{\mathrm{i}t}, 0 \leqslant t \leqslant \pi$, and observe that

$$\lim_{r\to 0} F_0(r\,\mathrm{e}^{\mathrm{i}t}) = \omega^2 + \lambda\beta,$$

which implies that on this semicircle

$$\lim_{r\to 0} \Delta \mathrm{Arg}\ F_0(s) = 0. \tag{31}$$

Now, according to the argument principle, relations (28)–(31) imply

$$N = \frac{1}{2\pi}\Delta \mathrm{Arg}\ F_0(s) = \frac{1}{2\pi} \times 2 \times 2\pi = 2,$$

where N denotes the number of zeros of F_0 in the domain.

Now we will prove that F_0 does not have zeros in the right half-plane. Consider the domain

$$\Omega^* = \{s \in \mathbb{C} : 0 < r < |s| < R, |\arg s| < \pi/2\}$$

and let $C^*_{r,R}$ be its boundary. Put $s = R\,\mathrm{e}^{\mathrm{i}t}, 0 \leqslant t \leqslant \frac{\pi}{2}$. Similarly, as in the first part, we conclude that

$$\lim_{R\to\infty} \Delta \mathrm{Arg}\ F_0(s) = \pi. \tag{32}$$

Put $s = t\,\mathrm{e}^{\mathrm{i}\pi/2}, r \leqslant t \leqslant R$. The imaginary part of $F_0(s)$ satisfies

$$\mathrm{Im}\ F_0(t\,\mathrm{e}^{\mathrm{i}\pi/2}) = \lambda\beta \frac{\frac{\pi}{2}(1+abt^2)\ln\frac{b}{a} + t(a-b)\left(\frac{\pi^2}{4} + \ln at \ln bt\right)}{(a^2t^2+1)\left(\frac{\pi^2}{4} + \ln^2 bt\right)} > 0. \tag{33}$$

We conclude that on this interval

$$\lim_{r\to 0, R\to\infty} \Delta \mathrm{Arg}\ F_0(s) = -\pi. \tag{34}$$

Using the argument principle, relation (28) and relations (29), (30) and (32) it follows

$$N = \frac{1}{2\pi} \times 2 \times (\pi - \pi) = 0.$$

This completes the proof. □

Remark 3. Positivity of (33) is equivalent to the positivity of

$$\phi(u,v) = \frac{\pi}{2}(1+uv)\ln\frac{v}{u} + (u-v)\left(\frac{\pi^2}{4} + \ln u \ln v\right), \qquad v > u > 0,$$

or

$$\psi_k(t) = \frac{\pi}{2}(1+kt^2)\ln k + t(1-k)\left(\frac{\pi^2}{4} + \ln t \ln kt\right), \qquad t > 0, k > 1.$$

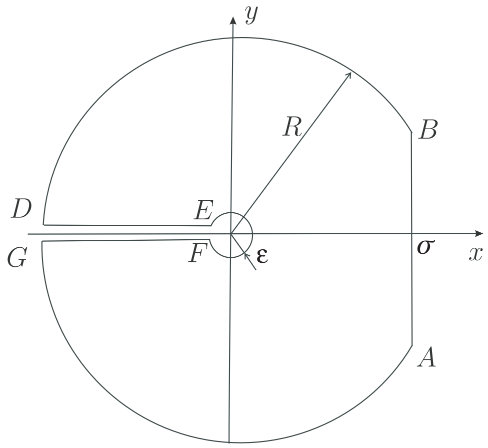

Figure 2. Integration contour γ_0.

3. Integral form of solutions

Recall,

$$\mathcal{L}y(s) = \frac{(y_0 s + v_0 + \widehat{h}(s))(\ln bs)(as-1)}{(s^2+\omega^2)(\ln bs)(as-1) + \lambda\beta(\ln as)(bs-1)}; \qquad \operatorname{Re} s > 0, \tag{35}$$

$$\mathcal{L}\sigma(s) = \lambda \frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1} \widehat{y}(s). \tag{36}$$

Applying the inverse Laplace transform to (35), we have

$$y(t) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{st} \mathcal{L}y(s)\,\mathrm{d}s, \qquad t \geqslant 0, \quad \text{where} \quad \gamma = \{s; \operatorname{Re} s = \sigma, \sigma > \sigma_0 = 0\}.$$

Let γ_0 be the contour on figure 2. Cauchy's formula gives

$$\int_{\gamma_0} \mathrm{e}^{st} \mathcal{L}y(s)\,\mathrm{d}s = 2\pi \mathrm{i} \sum \operatorname{Re} s\{\mathrm{e}^{st}\mathcal{L}y(s)\}. \tag{37}$$

Integrals

$$\int_{BD} \mathrm{e}^{st} \mathcal{L}y(s)\,\mathrm{d}s, \qquad \int_{GA} \mathrm{e}^{st} \mathcal{L}y(s)\,\mathrm{d}s \qquad \text{and} \qquad \int_{FE} \mathrm{e}^{st} \mathcal{L}y(s)\,\mathrm{d}s$$

tend to 0 when $R \to \infty$ and $\varepsilon \to 0$ so they do not contribute to the left-hand side of (37). What contribute to the left-hand side of (37) are integrals

$$\int_{AB} \mathrm{e}^{st} \mathcal{L}x(s)\,\mathrm{d}s, \qquad \int_{DE} \mathrm{e}^{st} \mathcal{L}x(s)\,\mathrm{d}s, \qquad \int_{FG} \mathrm{e}^{st} \mathcal{L}x(s)\,\mathrm{d}s.$$

We will show that when $r \to 0$ and $R \to \infty$ the contribution of the latter two integrals is provided by

$$2\pi \mathrm{i} F_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt} K_\alpha(r)\,\mathrm{d}r, \qquad t > 0,$$

where $K_\alpha(t)$ will be defined later, while the first integral tends to $\int_\gamma \mathrm{e}^{st}\mathcal{L}y(s)\,\mathrm{d}s$. Denote $G_\alpha(t) = \sum \operatorname{Re} s\{\mathrm{e}^{st}\mathcal{L}y(s)\}$, $t \geqslant 0$. Then,

$$\int_\gamma \mathrm{e}^{st} \mathcal{L}y(s)\,\mathrm{d}s + 2\pi \mathrm{i} F_\alpha(t) = 2\pi \mathrm{i} G_\alpha(t),$$

i.e.

$$y(t) = G_\alpha(t) - F_\alpha(t), \tag{38}$$

and we need to calculate F_α and G_α.

First we calculate $F_\alpha(t)$. Let $s = r\,\mathrm{e}^{\mathrm{i}\pi}$ on DE and $s = r\,\mathrm{e}^{-\mathrm{i}\pi}$ on FG, $|r| > 0$. Then

$$\int_{DE} \mathrm{e}^{st}\mathcal{L}y(s)\,\mathrm{d}s + \int_{FG} \mathrm{e}^{st}\mathcal{L}y(s)\,\mathrm{d}s$$
$$= \int_\varepsilon^R \mathrm{e}^{-tr}\frac{(\ln br + \mathrm{i}\pi)(-ar-1)(y_0(-r) + v_0 + \widehat{h}(-r))}{(r^2+\omega^2)(\ln br - \mathrm{i}\pi)(-ar-1) + \lambda\beta(\ln ar - \mathrm{i}\pi)(-br-1)}\,\mathrm{d}r$$
$$- \int_\varepsilon^R \mathrm{e}^{-tr}\frac{(\ln br - \mathrm{i}\pi)(-ar-1)(y_0(-r) + v_0 + \widehat{h}(-r))}{(r^2+\omega^2)(\ln br + \mathrm{i}\pi)(-ar-1) + \lambda\beta(\ln ar + \mathrm{i}\pi)(-br-1)}\,\mathrm{d}r.$$

Letting $R \to \infty$ and $\varepsilon \to 0$, it follows that

$$F_\alpha(t) = \beta\lambda \ln\frac{a}{b}\int_0^\infty \mathrm{e}^{-tr}\frac{A}{B}\,\mathrm{d}r, \tag{39}$$

where

$A = (ar+1)(br+1)[-y_0 r + v_0 + \widehat{h}(-r)]$,

$B = [(r^2+\omega^2)(ar+1)\ln br + \lambda\beta(br+1)\ln ar]^2 + \pi^2[(r^2+\omega^2)(ar+1) + \lambda\beta(br+1)]^2$.

Further, let $F_1(s) = y_0 s + v_0 + \widehat{h}(s)$. Then by (17) $\widehat{y}(s) = \frac{F_1(s)}{F(s)}$ and two simple zeros of $F(s)$ are the only singularities of $\mathrm{e}^{st}\widehat{y}(s)$. Thus,

$$\sum_{i=1}^2 \mathrm{Re}\,s\{\mathrm{e}^{st}\widehat{y}(s)\} = \sum_{i=1}^2 \frac{\mathrm{e}^{s_i t}F_1(s_i)}{F'(s_i)}. \tag{40}$$

Since $F(s) = s^2 + \omega^2 + \lambda\beta\frac{\ln(as)}{\ln(bs)}\frac{bs-1}{as-1}$ it follows that $F'(\overline{s}) = \overline{F'(s)}$. To write (40) in more explicit form, suppose that $\overline{\widehat{h}(s)} = \widehat{h}(\overline{s})$. Then $F_1(\overline{s}) = \overline{F_1(s)}$ and (40) may be written as

$$\sum_{i=1}^2 \mathrm{Re}\,s\{\mathrm{e}^{st}\widehat{y}(s)\} = \frac{2\mathrm{e}^{-\Sigma t}}{[\mathrm{Re}(F'(s_1))]^2 + [\mathrm{Im}(F'(s_1))]^2}$$
$$\times \{\mathrm{Re}[\overline{F_1(s_1)}F'(s_1)]\cos\Omega t + \mathrm{Im}[\overline{F_1(s_1)}F'(s_1)]\cos\Omega t\}, \tag{41}$$

where $s_{1,2} = \Sigma \pm \mathrm{i}\Omega$, $\Sigma < 0$, $\Omega > 0$. Therefore, the solution has the form

$$y(t) = \frac{2\mathrm{e}^{-\Sigma t}}{[\mathrm{Re}(F'(s_1))]^2 + [\mathrm{Im}(F'(s_1))]^2}\{\mathrm{Re}[\overline{F_1(s_1)}F'(s_1)]\cos\Omega t$$
$$+ \mathrm{Im}[\overline{F_1(s_1)}F'(s_1)]\cos\Omega t\} - \beta\lambda\ln\frac{a}{b}\int_0^\infty \mathrm{e}^{-tr}\frac{A(r)}{B(r)}\,\mathrm{d}r. \tag{42}$$

Remark 4. Equation (1) could be generalized by changing the integration limit so that

$$\int_0^2 \phi_\sigma(\gamma)\sigma^{(\gamma)}(x,t)\,\mathrm{d}\gamma = \int_0^2 \phi_\varepsilon(\gamma)\varepsilon^{(\gamma)}(x,t)\,\mathrm{d}\gamma. \tag{43}$$

The body described by (43) may be viewed as consisting of viscoelastic and viscoinertial elements [15]. If we take $\phi_\sigma(\gamma) = a^\gamma$, $\phi_\varepsilon(\gamma) = \Lambda b^\gamma$, then by applying the Laplace transform, instead of (18), we obtain

$$\widehat{F}(s) = s^2 + \omega^2 + \lambda\beta\frac{\ln(as)}{\ln(bs)}\frac{(bs)^2-1}{(as)^2-1}.$$

Also the explicit form of the constitutive equation, i.e., the solution of (43) with respect to σ, becomes

$$\sigma(t) = \sigma(t) = \lambda\widehat{\phi}_1 * \widehat{\phi}_2 * y(t),$$

where

$$\widehat{\phi}_1 = \mathcal{L}^{-1}\left(\frac{\ln as}{\ln bs}\right) = \delta(t) + \ln\frac{a}{b}\int_0^\infty \frac{t^{u-1}b^{-u}}{\Gamma(u)}\,\mathrm{d}u,$$
$$\phi_2 = \mathcal{L}^{-1}\left(\frac{(bs)^2-1}{(as)^2-1}\right) = \left(\frac{b}{a}\right)^2\delta(t) + \frac{b^2-a^2}{a^4}\sinh\frac{t}{a}.$$

The constitutive equation (43) is a generalization of relation (1) since (43) takes both viscoelastic and viscoinertial effects.

4. Conclusions

In this work, we studied vibrations of a single mass attached to a viscoelastic rod described by a fractional type, distributed-order constitutive equation and loaded with an arbitrary force $h(t)$. The motion of the system is described by the system of equations

$$y^{(2)}(t) + \omega^2 y + \beta\sigma(t) = h(t), \qquad \int_0^1 a^\gamma \sigma^{(\gamma)}(t)\,\mathrm{d}\gamma = \lambda\int_0^1 b^\gamma y^{(\gamma)}(t)\,\mathrm{d}\gamma, \tag{44}$$

subject to

$$y(0) = y_0, \qquad y^{(1)}(0) = v_0. \tag{45}$$

We showed for this system the following:

1. The solutions $y(t)$, $\sigma(t)$ exist and they are elements of $\mathcal{S}'_+$ and $\mathcal{K}'_+$, respectively. The form of $\sigma(t)$ is given, explicitly, as (27).
2. If the relaxation times a and b for stress and strain, respectively, satisfy the restrictions following from the second law of thermodynamics $0 < a < b$, we showed that the zeros of the function (18) are in the open left half-plane of the complex plane. This, together with the assumption that the forcing term $h(t)$ is equal to 0, implies that the solution has the form (42) representing viscously damped phase-shifted oscillation (41) superposed on the part that 'dies out' with time (39).
3. The stress–strain relation following from (13) is given by (27) and reads

$$\sigma(t) = \lambda\phi_1 * \phi_2 * y(t) = \lambda\left[\frac{b}{a}y(t) + \int_0^t \frac{b-a}{a^2}\,\mathrm{e}^{(t-\xi)/a}y(\xi)\,\mathrm{d}\xi\right.$$
$$+\frac{b}{a}\ln\frac{a}{b}\int_0^t y(\tau)\left(\int_0^\infty \frac{(t-\tau)^{u-1}b^{-u}}{\Gamma(u)}\,\mathrm{d}u\right)\mathrm{d}\tau$$
$$\left.+\frac{b-a}{a^2}\ln\frac{a}{b}\int_0^t \mathrm{e}^{(t-\xi)/a}y(\xi)\left(\int_0^\infty \frac{(t-\tau)^{u-1}b^{-u}}{\Gamma(u)}\,\mathrm{d}u\right)\mathrm{d}\tau\right]. \tag{46}$$

In the special case when $a = b$, the constitutive equation (13) describes an elastic body (see [4] p 690). By substituting $a = b$ into (46), we obtain

$$\sigma(t) = \lambda y(t), \tag{47}$$

i.e., the viscoelastic rod becomes an elastic rod.

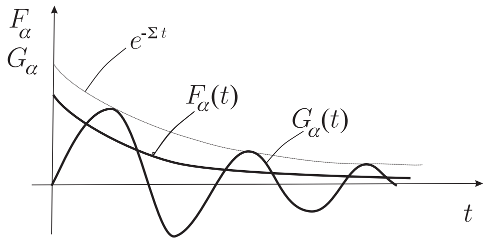

Figure 3. Qualitative properties of the functions $F_\alpha(t)$ and $G_\alpha(t)$.

4. The impulse response of the oscillator is obtained if we take $y_0 = v_0 = 0$, $h(t) = \delta(t)$ so that $\widehat{h}(r) = 1$. The function $y(t)$ in this case becomes
$$y(t) = G_\alpha(t) - F_\alpha(t),$$
where
$$G_\alpha(t) = \frac{2\mathrm{e}^{-\Sigma t}}{[\mathrm{Re}(F'(s_1))]^2 + [\mathrm{Im}(F'(s_1))]^2}\{\mathrm{Re}[\overline{F_1(s_1)}F'(s_1)]\cos\Omega t$$
$$+\,\mathrm{Im}[\overline{F_1(s_1)}F'(s_1)]\cos\Omega t\}; \tag{48}$$
$$F_\alpha(t) = -\beta\lambda\ln\frac{a}{b}\int_0^\infty \frac{\mathrm{e}^{-tr}(ar+1)(br+1)}{B}\,\mathrm{d}r.$$
Qualitatively the solution (48) is presented in figure 3. The function $y(t)$ could be used to obtain solution $y_h(t)$ for arbitrary $h(t)$ by forming the convolution $y(t) * h(t)$.
5. The solution of the distributed-order viscoelastic oscillator has the qualitative properties of the single-order oscillator studied in [18], by a different method.
It would be interesting to examine the solution in the case of a periodic forcing function, for both single and distributed-order fractional oscillators, as well as coupled systems of differential equations, treated in [25].
6. The results obtained here can be applied in linear viscoelasticity as well as in other areas where distributed-order differential equations arise. For equations of the types (12)–(14) in system identification theory, see, for example, [15].

Acknowledgment

We are grateful to the referee for helping us to improve an earlier version of the manuscript. This research was supported by the Ministry of Sciences of Serbia.

Appendix

Recall [16, 1], $\mathcal{K}(\mathbb{R})$ is the space of smooth functions ϕ with the property
$$\sup\{\mathrm{e}^{k|x|}|\phi^{(\alpha)}(x)|; x \in \mathbb{R}, \alpha \leqslant k\} < \infty, \qquad k \in \mathbb{N}_0. \tag{A.1}$$
Its dual is the space of exponentially bounded distributions $\mathcal{K}'(\mathbb{R})$ and elements of $\mathcal{K}'(\mathbb{R})$ are of the form $f = \sum_{\alpha=0}^{r}\Phi_\alpha^{(\alpha)}$, where Φ_α are continuous functions with the property $\Phi_\alpha(t) \leqslant C\,\mathrm{e}^{k_0|t|}, \alpha \leqslant r, t \in \mathbb{R}$, for some $C > 0, r \in \mathbb{N}_0$ and some $k_0 \in \mathbb{N}_0$. $\mathcal{K}'_+(\mathbb{R}) = \mathcal{K}'_+$ is a subspace of $\mathcal{K}'(\mathbb{R})$ consisting of elements supported by $[0, \infty)$; its elements are of the form
$$f = (\mathrm{e}^{kx}F(x))^{(p)}, \qquad x \in \mathbb{R}, \tag{A.2}$$

where F is a continuous bounded function such that $F(t) = 0, t \leqslant 0$. This implies that $f = (e^{kx}F(x))^{(p)}$, for some $F \in \mathcal{S}'_+$, $k \geqslant 0$ and some $p \in \mathbb{N}_0$.

Recall [24], that, if we take $(1 + x^2)^{k/2}$ instead of $e^{k|x|}$ in (A.1) we obtain well-known $\mathcal{S}(\mathbb{R})$ and related spaces $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}'_+$. Clearly, these spaces are subspaces of $\mathcal{K}'(\mathbb{R})$ and $\mathcal{K}'_+$, respectively.

The construction implies that elements of $\mathcal{K}'_+$ have the Laplace transformations, that is, if f is of the form (A.2), then its Laplace transform $\mathcal{L}f$ is an analytic function in the domain $\operatorname{Re} s > k$.

References

[1] Abdullah S 1999 *J. Math. Anal. Appl.* **233** 301
[2] Atanackovic T M 2002 *Continuum Mech. Thermodyn.* **14** 137
[3] Atanackovic T M 2002 *Acta Mech.* **159** 77
[4] Atanackovic T M 2003 *C. R. Acad. Sci., Paris II* **331** 687
[5] Atanackovic T M and Stankovic B 2002 *Z. Angew. Math. Mech.* **82** 377
[6] Bagely R L and Torvik P J 1986 *J. Rheol.* **30** 133
[7] Bagley R L and Torvik P J 2000 *Int. J. Appl. Math.* **2** 865
[8] Bagley R L and Torvik P J 2000 *Int. J. Appl. Math.* **2** 965
[9] Beyer H and Kempfle S 1995 *Z. Angew. Math. Mech.* **75** 623
[10] Caputo M and Mainardi F 1971 *Riv. Nuovo Cimento* **1** 161
[11] Caputo M 2001 *Fractional Calculus Appl. Anal.* **4** 421
[12] Enelund M, Mähler L, Runesson K and Lennart Josefson B 1999 *Int. J. Solids Struct.* **36** 2417
[13] Erdély A, Magnus W, Oberhettinger F and Tricomi F G 1954 *Tables of Integral Transforms* vol I (New York: McGraw-Hill)
[14] Gorenflo R and Mainardi F 1997 *Fractional Calculus in Continuum Mechanics* ed A Carpinteri and F Mainardi (Wien: Springer) p 223
[15] Hartely T T and Lorenzo C F 2003 *Signal Process.* **83** 2287
[16] Hasumi M 1961 *Tohoku Math. J.* **13** 94
[17] Mainardi F 1996 *Chaos Solitons Fractals* **7** 1461
[18] Narhari Achar B N, Hanneken J W, Enck T and Clarke T 2001 *Physica* A **297** 361
[19] Narhari Achar B N, Hanneken J W, Enck T and Clarke T 2002 *Physica* A **309** 275
[20] Pilipovic S and Stojanovic M 2004 *Math. Notes (Mat. Zametki)* **75** 135
[21] Rossikhin Yu and Shitikova M V 2001 *Z. Angew. Math. Mech.* **81** 363
[22] Samko S G, Kilbas A A and Marichev O I 1993 *Fractional Integrals and Derivatives* (Amsterdam: Gordon and Breach)
[23] Seredyńska M 2000 *J. Math. Phys.* **41** 2135
[24] Vladimirov V S 1973 *Generalized Functions in Mathematical Physics* (Moscow: Mir)
[25] Atanackovic T M and Stankovic B 2004 *J. Phys. A: Math. Gen.* **37** 1241